J Eng Math (2009) 64:219–235
DOI 10.1007/s10665-008-9254-y

Analytical decoupling of poroelasticity equations for acoustic-wave propagation and attenuation in a porous medium containing two immiscible fluids

Wei-Cheng Lo · Garrison Sposito · Ernest Majer

Received: 1 October 2007 / Accepted: 17 October 2008 / Published online: 12 November 2008

Abstract Poroelasticity theory has become an effective and accurate approach to analyzing the intricate mechanical behavior of a porous medium containing two immiscible fluids, a system encountered in many subsurface engineering applications. However, the resulting partial differential equations in the theory intrinsically take on a coupled form in the terms pertinent to inertial drag, viscous damping, and applied stress, making it difficult to obtain closed-form, steady-state analytical solutions to boundary-value problems except in special cases. In the present paper, we demonstrate that, for dilatational wave excitations, these partial differential equations can be decoupled analytically into three Helmholtz equations featuring complex-valued, frequency-dependent normal coordinates that correspond physically to three independent modes of dilatational wave motion. The normal coordinates in turn can be expressed in the frequency domain as three different linear combinations of the solid dilatation and the linearized increment of fluid content for each pore fluid, or equivalently, as three different linear combinations of total dilatational stress and two pore fluid pressures. These representations are applicable to strain-controlled and stress-prescribed boundary conditions, respectively. Numerical calculations confirm that the phase speed and attenuation coefficient of the three dilatational waves represented by the Helmholtz equations are exactly identical to those obtained previously by numerical solution of the dispersion relations for dilatational wave excitation of a porous medium containing two immiscible fluids. Thus, dilatational wave motions in unsaturated porous media subject to suitable boundary conditions can now be accurately modeled analytically.

Keywords Decoupling · Dilatational wave motions · Poroelasticity

1 Introduction

Strong coupling between applied stress and pore fluid pressure, termed poroelasticity [1], has long been noted in a wide range of phenomena encountered in hydrogeology, geomechanics, and reservoir engineering. Groundwater

W.-C. Lo (✉)
Department of Hydraulic and Ocean Engineering, National Cheng Kung University, Tainan 701, Taiwan, ROC
e-mail: lowc@mail.ncku.edu.tw

G. Sposito
Department of Civil and Environmental Engineering, University of California, Berkeley, CA 94720-1710, USA

G. Sposito · E. Majer
Department of Geophysics, Earth Sciences Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

withdrawal or hydrocarbon extraction can deform the solid matrix in a subsurface reservoir to trigger land subsidence [2–4]. The converse is true: deformation of porous media can cause pore volume changes that create additional pore pressure gradients, not only enhancing subsurface fluid flow [5, 6], but also altering surface stream discharge [7]. There is now a broad consensus that many hydrogeological and reservoir engineering problems can be understood well only if hydromechanical coupling between the solid skeleton and the interstitial fluids is taken into account [8–13].

A mathematical description of poroelasticity was first proposed by Biot [14] for the study of the consolidation problem in a homogeneous, isotropic, elastic porous medium permeated by a single compressible viscous fluid. A more general theoretical treatment of elastic-wave propagation and attenuation through a fluid-saturated porous medium was then developed in a famous series of papers by Biot [15, 16]. One fundamental feature of this theory is the prediction that a shear (rotational) wave together with two compressional (dilatational) waves exist in a poroelastic medium bearing a single fluid, as opposed to the existence of only a single compressional wave and a shear wave in a nonporous solid. The dilatational wave with the larger phase velocity, known as the Biot fast wave, propagates in a motional mode wherein the displacements of the solid and the fluid are always in phase, whereas the other dilatational wave, known as the Biot slow wave, is excited when the solid skeleton moves out of phase with the interstitial fluid [1, 15, 16].

When two immiscible pore fluids are present, the theoretical representation is more intricate because the fluids respond to both a pressure discontinuity (capillary pressure) and an acceleration difference (inertial coupling), neither of which is possible in a single-fluid system [17, Chaps. 6–8]. (A third response, to viscous coupling between two immiscible fluids caused by the difference in their velocities, is commonly ignored in models of two-phase flow through unsaturated porous media [18, Chap. 5].) Capillary pressure effects have been represented in most elastic-wave models based on either a Lagrangian [19, 20] or an Eulerian perspective [21–23], but inertial coupling has been typically ignored. The effect of inertial coupling was systematically evaluated by Berryman et al. [24], who neglected changes in capillary pressure, however, under the assumption that the excitation wavelength is sufficiently long to leave the pressure difference between two immiscible pore fluids effectively constant. Under this assumption, two coupled partial differential equations formally similar to the Biot [16] model equations can be derived and solved analytically, but with more complicated inertial and elasticity coefficients due to the presence of the second fluid [24].

Santos et al. [25], employing a Lagrangian variational principle, appear to have been the first to consider the joint influence of changes in capillary pressure and inertial coupling on poroelastic behavior. Recently, incorporation of these two phenomena into an Eulerian framework has been accomplished by Lo et al. [13] based on the continuum theory of mixtures. In both studies, the existence of three different compressional modes was demonstrated, designated conventionally as the P1, P2, and P3 waves in order of decreasing speed. The P1 and P2 waves are analogous to the fast and slow compressional waves in Biot theory. The P3 wave is associated with capillary pressure fluctuations, and has the highest attenuation coefficient with the lowest phase velocity [13, 22, 25].

Poroelasticity models feature partial differential equations that are coupled through physical terms describing inertial coupling, viscous damping, and applied stresses. A choice of normal coordinates for dilatational motions that allows exact separation of the coupled equations into three partial differential equations representing independent modes is highly desirable, particularly if each resulting equation is analytically solvable. For a porous medium containing a single fluid, exact decoupling of the Biot [16] model equations for dilatational motions can be achieved in the frequency domain [26, 27] using two complex-valued, frequency-dependent normal coordinates, each of which satisfies a Helmholtz equation (Table 1). In the time domain, Chandler and Johnson [28] demonstrated that, when inertial coupling terms are neglected in the Biot model, exact decoupling can be achieved with two real-valued normal coordinates that satisfy a diffusion equation and a Laplace equation, respectively. If the inertial coupling terms in the Biot model equations are retained, decoupling into the Chandler–Johnson normal coordinates is still possible, but requires a constraint relationship between elasticity coefficients and inertial coupling parameters [29]. However, an alternative condition has been found that does not require neglect of inertial coupling. Whenever the wave excitation frequency is much smaller than a critical frequency, defined by the ratio of the kinematic viscosity of the pore fluid to the permeability of the porous medium, the Biot model equations can be decoupled in the time

Table 1 Decoupled poroelasticity models

Fourier domain	One-fluid system	Two-fluid system	
		Capillary pressure changes neglected	Capillary pressure changes included
Models with inertial coupling			
Frequency	Dutta and Ode [26]; Berryman [27]	Berryman et al. [24]	Present study
Time (Low frequency limit)	Lo et al. [30]	Lo et al. [32]	Not available
	One-fluid system	Two-fluid system	
Models without inertial coupling			
Time	Chandler and Johnson [28]	Lo [31]	

domain, yielding a propagating-wave equation for the Biot fast wave and a dissipative wave equation for the Biot slow wave [30].

Decoupling of the Lo et al. [13] poroelasticity equations for dilatational waves in a porous medium containing two immiscible fluids also can be accomplished in the time domain if inertial coupling terms are dropped [31]. Berryman et al. [24] decoupled their model equations, which neglect capillary pressure changes but retain inertial coupling terms, following the frequency-domain method used by Berryman [27] for the single-fluid case (Table 1). In a generalization of the result for a single-fluid system [13], conversion of the decoupled frequency-domain equations of Berryman et al. [24] for dilatational waves into the time domain can be accomplished [32] if the wave excitation frequency is well below a critical frequency, equal to the ratio of an effective kinematic shear viscosity for the two interstitial fluids [24] to the intrinsic permeability of the porous medium (Table 1).

Since the Berryman et al. [24] model neglects changes in capillary pressure, only two decoupled frequency-domain equations are found. However, it has long been known that three modes of dilatational wave motion must exist in unsaturated porous media [13, 19, 21–23, 25]. Thus, a complete set of three decoupled Helmholtz equations is expected but heretofore has not been available. In the present paper, we derive these three decoupled Helmholtz equations. The dependent variables (normal coordinates) are three different linear combinations of the dilatation of the solid and the linearized increment of fluid content for each pore fluid [24], as developed in the dilatational wave model of Lo et al. [13] for unsaturated poroelastic media, which accounts for the effects of both changes in capillary pressure and inertial coupling. Based on linear stress–strain relationships among the solid and two fluid phases, the dependent variables can be converted into three different linear combinations of total dilatational stress and the two pore fluid pressures, corresponding to a stress-controlled scenario. A numerical study is then performed to show that our Helmholtz equations indeed yield the same phase velocities and attenuation coefficients for the dilatational waves as can be determined directly from conventional numerical solution of the dispersion relation for the coupled model equations. When specialized to a fully saturated elastic porous medium, our Helmholtz equations reduce to the two decoupled frequency-domain equations derived previously for the Biot poroelastic model [26, 27].

2 Poroelasticity model equations

Coupled partial differential equations describing dilatational wave propagation and attenuation through an elastic porous medium containing two immiscible, viscous, compressible fluids, allowing for changes in capillary pressure and inertial coupling, were developed by Lo et al. [13] in an Eulerian framework:

$$\rho_s\theta_s\frac{\partial^2 e}{\partial t^2}+A_{11}\left(\frac{\partial^2\varepsilon_1}{\partial t^2}-\frac{\partial^2 e}{\partial t^2}\right)+A_{12}\left(\frac{\partial^2\varepsilon_2}{\partial t^2}-\frac{\partial^2 e}{\partial t^2}\right)+A_{21}\left(\frac{\partial^2\varepsilon_1}{\partial t^2}-\frac{\partial^2 e}{\partial t^2}\right)+A_{22}\left(\frac{\partial^2\varepsilon_2}{\partial t^2}-\frac{\partial^2 e}{\partial t^2}\right)$$
$$+R_{11}\left(\frac{\partial\varepsilon_1}{\partial t}-\frac{\partial e}{\partial t}\right)+R_{22}\left(\frac{\partial\varepsilon_2}{\partial t}-\frac{\partial e}{\partial t}\right)=\left(a_{11}+\frac{4}{3}G\right)\nabla^2 e+a_{12}\nabla^2\varepsilon_1+a_{13}\nabla^2\varepsilon_2, \quad (1.1)$$

$$\rho_1\theta_1\frac{\partial^2\varepsilon_1}{\partial t^2}-A_{11}\left(\frac{\partial^2\varepsilon_1}{\partial t^2}-\frac{\partial^2 e}{\partial t^2}\right)-A_{12}\left(\frac{\partial^2\varepsilon_2}{\partial t^2}-\frac{\partial^2 e}{\partial t^2}\right)-R_{11}\left(\frac{\partial\varepsilon_1}{\partial t}-\frac{\partial e}{\partial t}\right)=a_{21}\nabla^2 e+a_{22}\nabla^2\varepsilon_1+a_{23}\nabla^2\varepsilon_2, \tag{1.2}$$

$$\rho_2\theta_2\frac{\partial^2\varepsilon_2}{\partial t^2}-A_{21}\left(\frac{\partial^2\varepsilon_1}{\partial t^2}-\frac{\partial^2 e}{\partial t^2}\right)-A_{22}\left(\frac{\partial^2\varepsilon_2}{\partial t^2}-\frac{\partial^2 e}{\partial t^2}\right)-R_{22}\left(\frac{\partial\varepsilon_2}{\partial t}-\frac{\partial e}{\partial t}\right)=a_{31}\nabla^2 e+a_{32}\nabla^2\varepsilon_1+a_{33}\nabla^2\varepsilon_2, \tag{1.3}$$

where ρ_α denotes the material density of phase α, the subscript α designating three immiscible phases: the solid, the nonwetting fluid ($\alpha = 1$; fluid 1), and the wetting fluid ($\alpha = 2$; fluid 2); θ_α signifies the volumetric fraction of phase α; e represents the volumetric strain (dilatation) of the solid phase; ε_ξ represents the volumetric strain (dilatation) of fluid phase ξ ($\xi = 1, 2$); R_{11} and R_{22} are the constitutive coefficients associated with viscous coupling between the solid and fluid phases; A_{11} and A_{22} are the constitutive coefficients pertinent to inertial coupling between the fluid and solid phases, while A_{12} and A_{21} are those concerning inertial coupling between the fluid phase and the adjacent fluid phase; G expresses the shear modulus of the porous framework; and a_{ij} ($i, j = 1, 2, 3$) are elasticity coefficients, and their cross terms are symmetric, i.e., $a_{ij} = a_{ji}$. The physical interpretation of the coefficients R_{11}, R_{22}, A_{11}, A_{22}, A_{12}, A_{21}, and a_{ij} is detailed in Lo et al. [13] and the representation of these coefficients in terms of directly-measurable parameters is given here in Appendix A.

By adding the equations of (1), one can obtain an equation that reflects dynamic equilibrium of the total stress for the entire three-phase system:

$$\rho_s\theta_s\frac{\partial^2 e}{\partial t^2}+\rho_1\theta_1\frac{\partial^2\varepsilon_1}{\partial t^2}+\rho_2\theta_2\frac{\partial^2\varepsilon_2}{\partial t^2}=(\widetilde{a}_{11}+a_{21}+a_{31})\nabla^2 e+(a_{12}+a_{22}+a_{32})\nabla^2\varepsilon_1+(a_{13}+a_{23}+a_{33})\nabla^2\varepsilon_2, \tag{2}$$

where $\widetilde{a}_{11} = a_{11} + \frac{4}{3}G$. We note that, according to the theory of poroelasticity, an elastic porous medium undergoes very small deformations in a reversible thermodynamic process, so all physical parameters ρ_α, θ_α, a_{ij}, A_{11}, A_{12}, A_{21}, A_{22}, R_{11}, and R_{22} in (1.2), (1.3), and (2) are evaluated in a reference configuration, taken here as the unperturbed state before stress is applied [13, 16, 22, 24, 25, 33, 34].

3 Decoupling the model equations

The Biot model equations for dilatational wave motions in a porous medium containing a single fluid can be decoupled exactly after Fourier time-transformation of the dilatation of the solid and the linearized increment of fluid content into the frequency domain [26, 27]. The same dependent variables and Fourier transformation approach were used by Berryman et al. [24] to decouple their model equations for an elastic porous medium containing two fluids under the assumption of negligible capillary pressure change. Similarly, we employ the linearized increment of fluid content for each pore fluid, $\zeta_\xi = \theta_\xi(\varepsilon_\xi - e)$ ($\xi = 1, 2$) defined by Berryman et al. [24], as the dependent variable instead of ε_ξ in equations (1.2), (1.3), and (2):

$$\rho_1\frac{\partial^2 e}{\partial t^2}+\left[\frac{(\rho_1\theta_1-A_{11})}{\theta_1^2}\right]\frac{\partial^2\zeta_1}{\partial t^2}-\frac{A_{12}}{\theta_1\theta_2}\frac{\partial^2\zeta_2}{\partial t^2}-\frac{R_{11}}{\theta_1^2}\frac{\partial\zeta_1}{\partial t}=\frac{(a_{12}+a_{22}+a_{23})}{\theta_1}\nabla^2 e+\frac{a_{22}}{\theta_1^2}\nabla^2\zeta_1+\frac{a_{23}}{\theta_1\theta_2}\nabla^2\zeta_2, \tag{3.1}$$

$$\rho_2\frac{\partial^2 e}{\partial t^2}+\left[\frac{(\rho_2\theta_2-A_{22})}{\theta_2^2}\right]\frac{\partial^2\zeta_2}{\partial t^2}-\frac{A_{12}}{\theta_1\theta_2}\frac{\partial^2\zeta_1}{\partial t^2}-\frac{R_{22}}{\theta_2^2}\frac{\partial\zeta_2}{\partial t}=\frac{(a_{13}+a_{23}+a_{33})}{\theta_2}\nabla^2 e+\frac{a_{23}}{\theta_1\theta_2}\nabla^2\zeta_1+\frac{a_{33}}{\theta_2^2}\nabla^2\zeta_2, \tag{3.2}$$

$$(\rho_s\theta_s + \rho_1\theta_1 + \rho_2\theta_2)\frac{\partial^2 e}{\partial t^2} + \rho_1\frac{\partial^2 \zeta_1}{\partial t^2} + \rho_2\frac{\partial^2 \zeta_2}{\partial t^2} = a_T\nabla^2 e + \frac{(a_{12}+a_{22}+a_{23})}{\theta_1}\nabla^2\zeta_1 + \frac{(a_{13}+a_{23}+a_{33})}{\theta_2}\nabla^2\zeta_2, \tag{3.3}$$

where $a_T = \widetilde{a}_{11} + a_{22} + a_{33} + 2(a_{12} + a_{13} + a_{23})$. Definition of

$$\chi_\xi \equiv -\frac{R_{\xi\xi}}{\theta_\xi^2}, \quad (\xi = 1, 2) \tag{4.1}$$

$$\rho_{\xi s} \equiv \frac{\rho_\xi\theta_\xi - A_{\xi\xi}}{\theta_\xi^2}, \quad (\xi = 1, 2) \tag{4.2}$$

$$\rho_{12} \equiv \frac{A_{12}}{\theta_1\theta_2}, \tag{4.3}$$

$$\rho \equiv \sum \rho_\alpha\theta_\alpha, \tag{4.4}$$

enables us to express the equations (3) as:

$$\rho\frac{\partial^2 e}{\partial t^2} + \rho_1\frac{\partial^2 \zeta_1}{\partial t^2} + \rho_2\frac{\partial^2 \zeta_2}{\partial t^2} = D_{11}\nabla^2 e + D_{12}\nabla^2\zeta_1 + D_{13}\nabla^2\zeta_2, \tag{5.1}$$

$$\rho_1\frac{\partial^2 e}{\partial t^2} + \rho_{1s}\frac{\partial^2 \zeta_1}{\partial t^2} - \rho_{12}\frac{\partial^2 \zeta_2}{\partial t^2} + \chi_1\frac{\partial \zeta_1}{\partial t} = D_{12}\nabla^2 e + D_{22}\nabla^2\zeta_1 + D_{23}\nabla^2\zeta_2, \tag{5.2}$$

$$\rho_2\frac{\partial^2 e}{\partial t^2} - \rho_{12}\frac{\partial^2 \zeta_1}{\partial t^2} + \rho_{2s}\frac{\partial^2 \zeta_2}{\partial t^2} + \chi_2\frac{\partial \zeta_2}{\partial t} = D_{13}\nabla^2 e + D_{23}\nabla^2\zeta_1 + D_{33}\nabla^2\zeta_2. \tag{5.3}$$

The elasticity coefficients D_{ij} defined in (5) are given by

$$D_{11} = a_T, \quad D_{12} = \frac{(a_{12}+a_{22}+a_{23})}{\theta_1}, \quad D_{13} = \frac{(a_{13}+a_{23}+a_{33})}{\theta_2}, \tag{6.1, 2, 3}$$

$$D_{22} = \frac{a_{22}}{\theta_1^2}, \quad D_{23} = \frac{a_{23}}{\theta_1\theta_2}, \quad D_{33} = \frac{a_{33}}{\theta_2^2}. \tag{6.4, 5, 6}$$

The parameters χ_ξ, $\rho_{\xi s}$, and ρ_{12} in (4.1)–(4.3) can be written in a more tractable way by substituting (A4) and (A5), respectively:

$$\chi_\xi = \frac{\eta_\xi}{k_s k_{r\xi}}, \quad (\xi = 1, 2) \tag{7.1}$$

$$\rho_{\xi s} = \frac{\alpha_s\rho_\xi}{\theta_\xi}, \quad (\xi = 1, 2) \tag{7.2}$$

$$\rho_{12} = -0.1\sqrt{\frac{\alpha_s^2\rho_1\rho_2}{\theta_1\theta_2}}, \tag{7.3}$$

where χ_ξ and $\rho_{\xi s}$ are analogous to the coefficients b and m defined by Biot [16] to represent viscous and inertial coupling, respectively, in a single-fluid system. In (7), η_ξ is the dynamic shear viscosity of fluid phase ξ; k_s is the intrinsic permeability of the porous medium; $k_{r\xi}$ is the relative permeability of fluid phase ξ; and α_s is the tortuosity of the porous medium.

After Fourier time-transformation $[f(\vec{x}, t) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(\vec{x}, \omega)\exp(-\mathrm{i}\omega t)\mathrm{d}\omega]$, we can then recast the equations (5) as

$$\left\{\begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix}\nabla^2 + \omega^2\begin{bmatrix} \rho & \rho_1 & \rho_2 \\ \rho_1 & q_1 & -\rho_{12} \\ \rho_2 & -\rho_{12} & q_2 \end{bmatrix}\right\}\begin{bmatrix} \tilde{e} \\ \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \end{bmatrix} = 0, \tag{8}$$

where $q_1 = \rho_{1s} + \frac{\mathrm{i}}{\omega}\chi_1$ and $q_2 = \rho_{2s} + \frac{\mathrm{i}}{\omega}\chi_2$. Next, inverting the first matrix in (8), we have

$$\left\{\nabla^2 + \frac{\omega^2}{\Delta}\begin{bmatrix} D_{22}D_{33} - D_{23}^2 & D_{13}D_{23} - D_{12}D_{33} & D_{12}D_{23} - D_{13}D_{22} \\ D_{13}D_{23} - D_{12}D_{33} & D_{11}D_{33} - D_{13}^2 & D_{12}D_{13} - D_{11}D_{23} \\ D_{12}D_{23} - D_{13}D_{22} & D_{12}D_{13} - D_{11}D_{23} & D_{11}D_{22} - D_{12}^2 \end{bmatrix} \times \begin{bmatrix} \rho & \rho_1 & \rho_2 \\ \rho_1 & q_1 & -\rho_{12} \\ \rho_2 & -\rho_{12} & q_2 \end{bmatrix}\right\}\begin{bmatrix} \tilde{e} \\ \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \end{bmatrix} = 0, \tag{9}$$

where $\Delta = D_{11}D_{22}D_{33} + 2D_{12}D_{13}D_{23} - D_{13}^2 D_{22} - D_{11}D_{23}^2 - D_{12}^2 D_{33}$. Equation (9) can be compactly arranged into matrix form:

$$(\overline{\overline{\delta}}\nabla^2 + \overline{\overline{B}})\begin{bmatrix} \tilde{e} \\ \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \end{bmatrix} = 0, \tag{10}$$

where $\overline{\overline{\delta}}$ is the unit tensor ($\delta_{ij} = 0$ for $i \neq j$, $\delta_{ij} = 1$ for $i = j$) and $\overline{\overline{B}} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$ with its matrix elements being:

$$B_{11} = \omega^2[(D_{22}D_{33} - D_{23}^2)\rho + (D_{13}D_{23} - D_{12}D_{33})\rho_1 + (D_{12}D_{23} - D_{13}D_{22})\rho_2]/\Delta, \tag{11.1}$$

$$B_{12} = \omega^2[(D_{22}D_{33} - D_{23}^2)\rho_1 + (D_{13}D_{23} - D_{12}D_{33})q_1 - (D_{12}D_{23} - D_{13}D_{22})\rho_{12}]/\Delta, \tag{11.2}$$

$$B_{13} = \omega^2[(D_{22}D_{33} - D_{23}^2)\rho_2 - (D_{13}D_{23} - D_{12}D_{33})\rho_{12} + (D_{12}D_{23} - D_{13}D_{22})q_2]/\Delta, \tag{11.3}$$

$$B_{21} = \omega^2[(D_{13}D_{23} - D_{12}D_{33})\rho + (D_{11}D_{33} - D_{13}^2)\rho_1 + (D_{12}D_{13} - D_{11}D_{23})\rho_2]/\Delta, \tag{11.4}$$

$$B_{22} = \omega^2[(D_{13}D_{23} - D_{12}D_{33})\rho_1 + (D_{11}D_{33} - D_{13}^2)q_1 - (D_{12}D_{13} - D_{11}D_{23})\rho_{12}]/\Delta, \tag{11.5}$$

$$B_{23} = \omega^2[(D_{13}D_{23} - D_{12}D_{33})\rho_2 - (D_{11}D_{33} - D_{13}^2)\rho_{12} + (D_{12}D_{13} - D_{11}D_{23})q_2]/\Delta, \tag{11.6}$$

$$B_{31} = \omega^2[(D_{12}D_{23} - D_{13}D_{22})\rho + (D_{12}D_{13} - D_{11}D_{23})\rho_1 + (D_{11}D_{22} - D_{12}^2)\rho_2]/\Delta, \tag{11.7}$$

$$B_{32} = \omega^2[(D_{12}D_{23} - D_{13}D_{22})\rho_1 + (D_{12}D_{13} - D_{11}D_{23})q_1 - (D_{11}D_{22} - D_{12}^2)\rho_{12}]]/\Delta, \tag{11.8}$$

$$B_{33} = \omega^2[(D_{12}D_{23} - D_{13}D_{22})\rho_2 - (D_{12}D_{13} - D_{11}D_{23})\rho_{12} + (D_{11}D_{22} - D_{12}^2)q_2]/\Delta. \tag{11.9}$$

Decoupling of the equations (10) corresponds to solving an eigenvalue problem for the matrix of $\overline{\overline{B}}$. The resulting decoupled equations are of the form:

$$[\nabla^2 + \lambda_j(\omega)]\Phi_j(\vec{x}, \omega) = 0, \tag{12}$$

where $\Phi_j(\vec{x}, \omega) = \Gamma_j\tilde{e} + \tilde{\zeta}_1 + \Pi_j\tilde{\zeta}_2$ ($j = 1, 2, 3$) is an eigenvector and $\lambda_j(\omega)$ is the corresponding eigenvalue that satisfies the cubic polynomial equation:

$$\begin{aligned}\lambda_j^3 &- (B_{11} + B_{22} + B_{33})\lambda_j^2 - (B_{13}B_{31} + B_{23}B_{32} + B_{12}B_{21} - B_{11}B_{33} - B_{22}B_{33} - B_{11}B_{22})\lambda_j \\ &+ (B_{13}B_{22}B_{31} + B_{11}B_{23}B_{32} + B_{12}B_{21}B_{33} - B_{11}B_{22}B_{33} - B_{12}B_{23}B_{31} - B_{13}B_{21}B_{32}) = 0,\end{aligned} \tag{13}$$

subject to three well-known constraints resulting from scalar invariance under similarity transformations:

$$\lambda_1 + \lambda_2 + \lambda_3 = (B_{11} + B_{22} + B_{33}), \tag{14.1}$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = -(B_{13}B_{31} + B_{23}B_{32} + B_{12}B_{21} - B_{11}B_{33} - B_{22}B_{33} - B_{11}B_{22}), \tag{14.2}$$

$$\lambda_1\lambda_2\lambda_3 = -(B_{13}B_{22}B_{31} + B_{11}B_{23}B_{32} + B_{12}B_{21}B_{33} - B_{11}B_{22}B_{33} - B_{12}B_{23}B_{31} - B_{13}B_{21}B_{32}). \tag{14.3}$$

The parameters Γ_j and Π_j are related to the eigenvalue $\lambda_j(\omega)$ by

$$\lambda_j(\omega) = B_{11} + \frac{B_{21}}{\Gamma_j} + \frac{\Pi_j}{\Gamma_j} B_{31} = \Gamma_j B_{12} + B_{22} + \Pi_j B_{32} = \frac{\Gamma_j}{\Pi_j} B_{13} + \frac{B_{23}}{\Pi_j} + B_{33}. \tag{15}$$

Equation (15) can be solved to yield

$$\Gamma_j = \frac{(\lambda_j B_{31} + B_{21}B_{32} - B_{22}B_{31})}{(\lambda_j B_{32} + B_{12}B_{31} - B_{11}B_{32})}, \quad \Pi_j = \frac{(\lambda_j B_{13} + B_{12}B_{23} - B_{13}B_{22})}{(\lambda_j B_{12} + B_{13}B_{32} - B_{12}B_{33})}. \tag{16.1, 2}$$

Physically, $\sqrt{\lambda}_j = k_j = k_j^r + \mathrm{i}k_j^i$ $(j = 1, 2, 3)$ represents a complex wavenumber that includes attenuation, where $k_j^r = \mathfrak{Re}(k_j)$ expresses the conventional wavenumber and $k_j^i = \mathfrak{Im}(k_j)$ denotes the attenuation coefficient [13]. Accordingly, the phase speed $\overline{V}_j$ can be determined through the standard relation: $\overline{V}_j = \frac{\omega}{k_j^r}$. The amplitude of dilatational waves decays with distance; in turn, this requires $k_j^i > 0$ mathematically. Therefore, the resulting Helmholtz equations (12) provide three dilatational wave solutions. If the dilatational waves are steady-state, harmonic, and travel along the z-direction (plane waves), the solution to (12) must take the general form:

$$\Phi_1(z,\omega) = \beta_1 \exp\mathrm{i}(\sqrt{\lambda_1}z - \omega t), \quad \Phi_2(z,\omega) = \beta_2 \exp\mathrm{i}(\sqrt{\lambda_2}z - \omega t), \quad \Phi_3(z,\omega) = \beta_3 \exp\mathrm{i}(\sqrt{\lambda_3}z - \omega t), \tag{17.1, 2, 3}$$

where β_1, β_2, and β_3 are wave amplitudes to be determined from the boundary conditions; λ_1, λ_2, and λ_3 are obtained by solving (13) and $\mathfrak{Re}(\sqrt{\lambda_1}) < \mathfrak{Re}(\sqrt{\lambda_2}) < \mathfrak{Re}(\sqrt{\lambda_3})$ is defined. As a result, the analytical solutions for the dilatation of the solid and the linearized increment of fluid content of fluid phase ξ in the frequency domain is found to be:

$$\widetilde{e} = [(\Pi_3 - \Pi_2)\Phi_1 + (\Pi_1 - \Pi_3)\Phi_2 + (\Pi_2 - \Pi_1)\Phi_3]/\Psi, \tag{18.1}$$

$$\widetilde{\zeta}_1 = [(\Gamma_3\Pi_2 - \Gamma_2\Pi_3)\Phi_1 + (\Gamma_1\Pi_3 - \Gamma_3\Pi_1)\Phi_2 + (\Gamma_2\Pi_1 - \Gamma_1\Pi_2)\Phi_3]/\Psi, \tag{18.2}$$

$$\widetilde{\zeta}_2 = [(\Gamma_2 - \Gamma_3)\Phi_1 + (\Gamma_3 - \Gamma_1)\Phi_2 + (\Gamma_1 - \Gamma_2)\Phi_3]/\Psi, \tag{18.3}$$

where $\Psi = \Gamma_1\Pi_3 + \Gamma_2\Pi_1 + \Gamma_3\Pi_2 - \Gamma_3\Pi_1 - \Gamma_1\Pi_2 - \Gamma_2\Pi_3$. The values of the parameters Γ_j and Π_j corresponding to each eigenvalue λ_j are calculated from (16). The Helmholtz equations (12) that result from decoupling (1.2), (1.3), and (2) can be applied to model analytically the behavior of dilatational waves in unsaturated porous media under a variety of boundary conditions. Lastly, we note in passing that an alternative approach to ours for solving the Helmholtz equations (12) involves the methods of matrix differential calculus [35].

4 Numerical verification

To verify that the Helmholtz equations (12) correctly represent the independent motional modes of three dilatational waves, a numerical calculation was conducted to determine their phase speeds and attenuation coefficients in Columbia fine sandy loam containing either an oil–water or an air–water mixture, two illustrative examples previously studied by Lo et al. [13]. The well-known van Genuchten [36]–Mualem [37] model was applied to represent the capillary pressure—fluid saturation and relative permeability—fluid saturation relationships required for computation of the coefficients a_{ij} and χ_ξ. The elasticity and hydraulic data, together with the fitting parameters in the van Genuchten–Mualem model necessary for numerical simulation, were taken from Lo et al. [13]. The cubic polynomial equation (13) was numerically solved in MATLAB to determine the eigenvalues λ_i. Figures 1–6 show the phase speeds and attenuation coefficients of the three dilatational waves in both fluid mixtures as functions of excitation frsequency and water saturation. Comparison of Figs. 1–6 with the results in [13] demonstrates that the phase speeds of the three dilatational waves determined from the Helmholtz equations (12) are entirely consistent with those predicted from conventional numerical solution of the dispersion relation for the coupled model equations (1) reported in [13]. This consistency also holds true for the attenuation coefficients, confirming that the

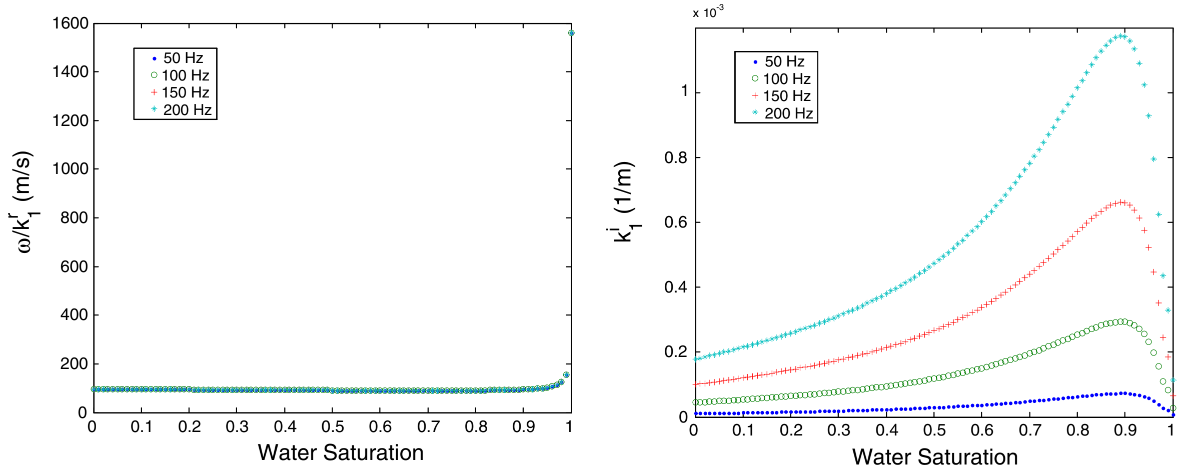

Fig. 1 The phase speed and attenuation coefficient of the P1 wave as a function of excitation frequency and water saturation in an air–water system

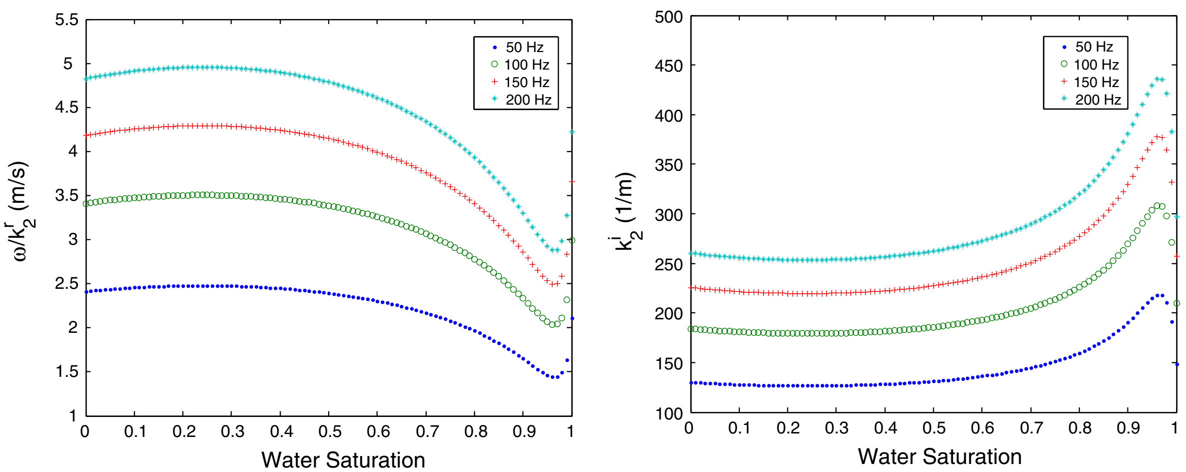

Fig. 2 The phase speed and attenuation coefficient of the P2 wave as a function of excitation frequency and water saturation in an air–water system

Helmholtz equations (12) indeed provide an accurate description of three uncoupled motional modes of dilatational wave motions in an unsaturated porous medium. The maximum relative difference obtained for the phase speeds and attenuation coefficients that were computed from these two solutions was found to be less than 10^{-6}%. A comprehensive discussion of the important physical parameters controlling the phase speeds and attenuation coefficients of these waves was given by Lo et al. [13].

5 Alternative normal coordinates

The boundary conditions for poroelasticity problems encountered in hydrogeology [9, Chaps. 6–9], and soil dynamics [38, Chaps. 3–4], are commonly specified in terms of applied stress instead of induced strain. Evidently the primary dependent variables in (12) are expressed in terms of the displacement components (dilatations) of the solid and fluids. If a boundary-value problem involving (12) is established under stress boundary conditions, these dependent variables must be converted into linear combinations of the applied stress.

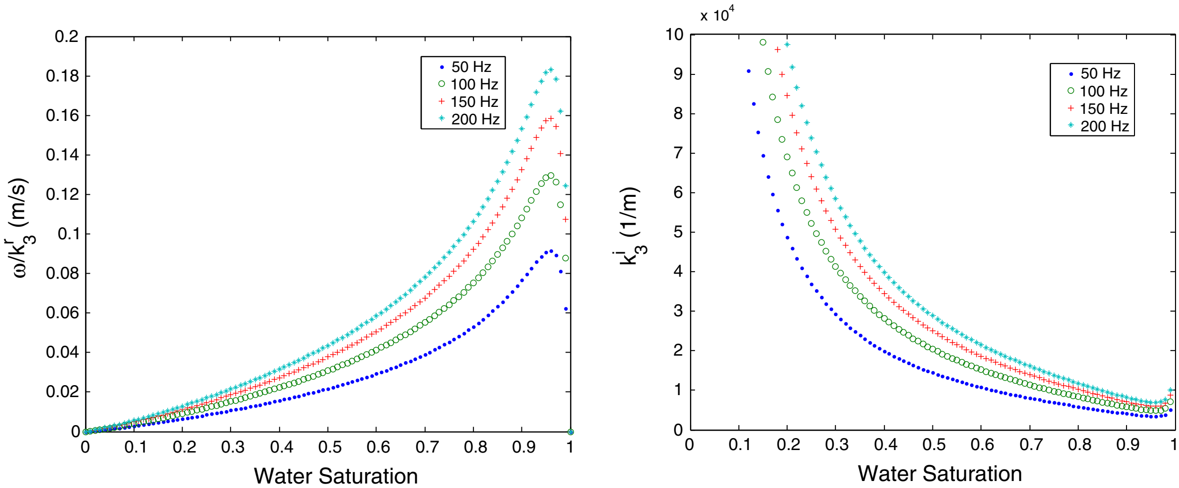

Fig. 3 The phase speed and attenuation coefficient of the P3 wave as a function of excitation frequency and water saturation in an air–water system

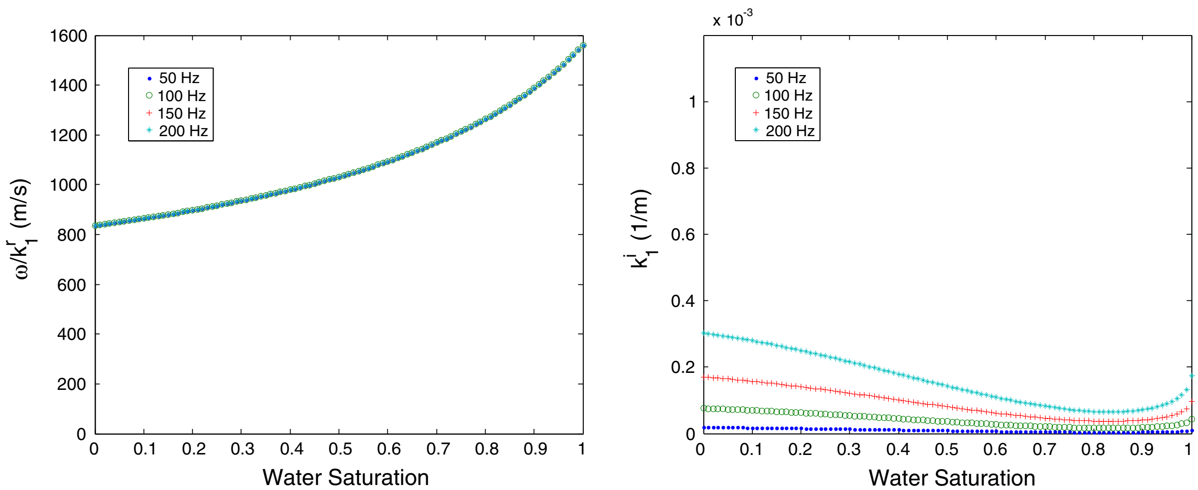

Fig. 4 The phase speed and attenuation coefficient of the P1 wave as a function of excitation frequency and water saturation in an oil–water system

According to the conventional definition of the total stress of the bulk porous material [16], the total stress $\overline{\overline{\sigma}}$ placed on a porous medium containing a single fluid is borne in part by the stress applied to the solid, $\overline{\overline{t}}_s$, and in part by the force per unit area acting on the fluid, $-\phi p_f$, where p_f represents the (gauge) pore fluid pressure [16]:

$$\overline{\overline{\sigma}} = \overline{\overline{t}}_s - \phi p_f \overline{\overline{\delta}}. \tag{19}$$

Using a variational principle, Biot [15, 16] developed linear stress–strain relationships for an elastic porous medium bearing a single compressible fluid based on the strain energy function. Lo et al. [13] generalized the Biot [15] linear stress–strain relationships to apply to a two-fluid system:

$$\overline{\overline{t}}_s = 2G\overline{\overline{e}} + \left[(a_{11} - \frac{2}{3}G)e + a_{12}\varepsilon_1 + a_{13}\varepsilon_2\right]\overline{\overline{\delta}}, \tag{20.1}$$

$$-\theta_1 p_1 = a_{12}e + a_{22}\varepsilon_1 + a_{23}\varepsilon_2, \quad -\theta_2 p_2 = a_{13}e + a_{23}\varepsilon_1 + a_{33}\varepsilon_2, \tag{20.2, 3}$$

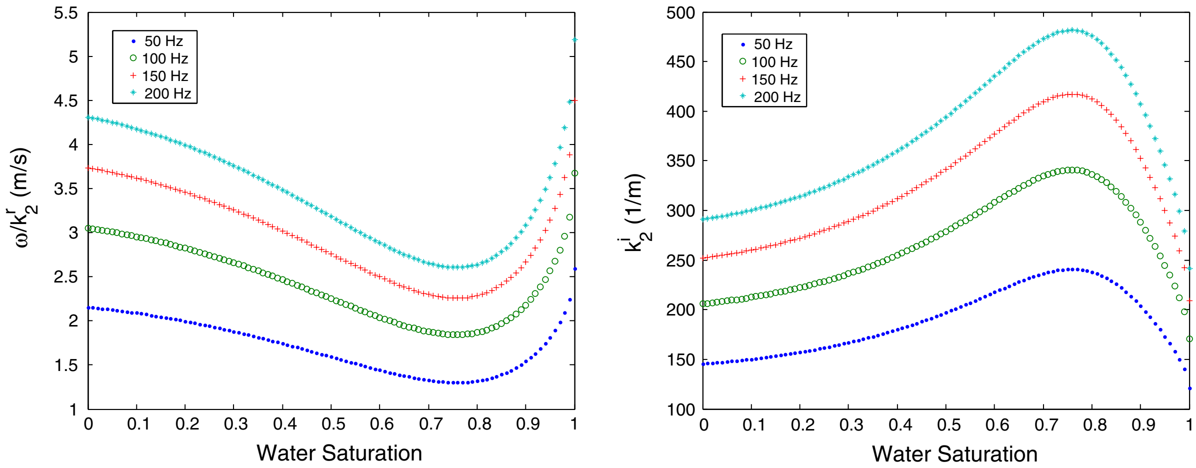

Fig. 5 The phase speed and attenuation coefficient of the P2 wave as a function of excitation frequency and water saturation in an oil–water system

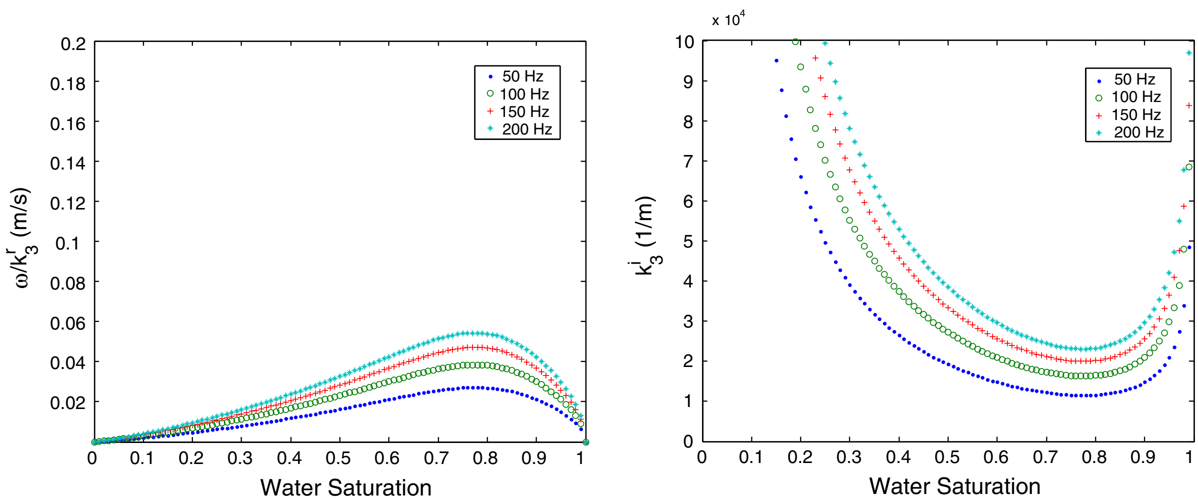

Fig. 6 The phase speed and attenuation coefficient of the P3 wave as a function of excitation frequency and water saturation in an oil–water system

where p_ξ ($\xi = 1, 2$) refers to the (gauge) pressure in each fluid; $\bar{\bar{e}} = \frac{1}{2}(\vec{\nabla}\vec{u}_s + \vec{\nabla}\vec{u}_s^T)$ signifies the solid strain tensor, the superscript T representing its transpose; and $\vec{u}_s$ denotes the displacement vector of the solid phase.

In a two-fluid system, each fluid typically experiences a different pore pressure, so the fluid pressure p_f defined in (19) is generalized by considering it as an average pressure in a mixture comprising the non-wetting fluid and the wetting fluid [13, 22]. This is also a common approach used to define the effective stress of an elastic porous medium saturated by two immiscible fluids [39], [40, Sect. 2.5.2]. Employing the technique of volume-averaging, Whitaker [41] showed that the average fluid pressure in such a porous medium is the weighted sum of the pressures in the nonwetting and wetting fluids:

$$p_f = S_1 p_1 + (1 - S_1) p_2. \tag{21}$$

Incorporation of (21) into (19) yields

$$\bar{\bar{\sigma}} = \bar{\bar{t}}_s - \phi[S_1 p_1 + (1 - S_1) p_2]\bar{\bar{\delta}} = \bar{\bar{t}}_s - (\theta_1 p_1 + \theta_2 p_2)\bar{\bar{\delta}}. \tag{22}$$

The diagonal component of the total stress tensor is the total dilatational stress σ_{kk} [28]:

$$\sigma_{kk} = t_s^{xx} - (\theta_1 p_1 + \theta_2 p_2) + t_s^{yy} - (\theta_1 p_1 + \theta_2 p_2) + t_s^{zz} - (\theta_1 p_1 + \theta_2 p_2), \tag{23}$$

where t_s^{ii} is a Cartesian element of $\bar{\bar{t}}_s$. By use of the linear stress–strain relationships formulated in (20) to replace the stress terms on the right side of (23), one can obtain

$$\sigma_{kk} = 3(a_{11} + a_{12} + a_{13})e + 3(a_{12} + a_{22} + a_{23})\varepsilon_1 + 3(a_{13} + a_{23} + a_{33})\varepsilon_2. \tag{24}$$

In view of (20.2), (20.3), and (24), the dilatations of the solid and two fluid phases can be written in terms of p_1, p_2, and σ_{kk}:

$$e = C_{11}\sigma_{kk} + C_{12}(-\theta_1 p_1) + C_{13}(-\theta_2 p_2), \quad \varepsilon_1 = C_{21}\sigma_{kk} + C_{22}(-\theta_1 p_1) + C_{23}(-\theta_2 p_2),$$
$$\varepsilon_2 = C_{31}\sigma_{kk} + C_{32}(-\theta_1 p_1) + C_{33}(-\theta_2 p_2), \tag{25.1, 2, 3}$$

where the elements of the matrix C_{ij} are given by

$$C_{11} = (a_{22}a_{33} - a_{23}^2)/\Lambda, \tag{26.1}$$

$$C_{12} = [3(a_{13} + a_{23} + a_{33})a_{23} - 3(a_{12} + a_{22} + a_{23})a_{33}]/\Lambda, \tag{26.2}$$

$$C_{13} = [3(a_{12} + a_{22} + a_{23})a_{23} - 3(a_{13} + a_{23} + a_{33})a_{22}]/\Lambda, \tag{26.3}$$

$$C_{21} = (a_{13}a_{23} - a_{12}a_{33})/\Lambda, \tag{26.4}$$

$$C_{22} = [3(a_{11} + a_{12} + a_{13})a_{33} - 3(a_{13} + a_{23} + a_{33})a_{13}]/\Lambda, \tag{26.5}$$

$$C_{23} = [3(a_{13} + a_{23} + a_{33})a_{12} - 3(a_{11} + a_{12} + a_{13})a_{23}]/\Lambda, \tag{26.6}$$

$$C_{31} = (a_{12}a_{23} - a_{13}a_{22})/\Lambda, \tag{26.7}$$

$$C_{32} = [3(a_{12} + a_{22} + a_{23})a_{13} - 3(a_{11} + a_{12} + a_{13})a_{23}]/\Lambda, \tag{26.8}$$

$$C_{33} = [3(a_{11} + a_{12} + a_{13})a_{22} - 3(a_{12} + a_{22} + a_{23})a_{12}]/\Lambda, \tag{26.9}$$

$$\Lambda = 3(a_{11} + a_{12} + a_{13})a_{22}a_{33} + 3(a_{12} + a_{22} + a_{23})a_{13}a_{23} + 3(a_{13} + a_{23} + a_{33})a_{12}a_{23}$$
$$- 3(a_{11} + a_{12} + a_{13})a_{23}^2 - 3(a_{12} + a_{22} + a_{23})a_{12}a_{33} - 3(a_{13} + a_{23} + a_{33})a_{22}a_{13}. \tag{26.10}$$

It follows from (25) that the linearized increment of fluid content for each pore fluid can be expressed using total dilatational stress and two pore fluid pressures:

$$\zeta_1 = \theta_1(C_{21} - C_{11})\sigma_{kk} + \theta_1^2(C_{12} - C_{22})p_1 + \theta_1\theta_2(C_{13} - C_{23})p_2, \tag{27.1}$$

$$\zeta_2 = \theta_2(C_{31} - C_{11})\sigma_{kk} + \theta_1\theta_2(C_{12} - C_{32})p_1 + \theta_2^2(C_{13} - C_{33})p_2. \tag{27.2}$$

Thus, after inserting equations (25.1) and (27) into the Helmholtz equations (12) and taking Fourier transformations, we derive equivalent Helmholtz equations under a stress scenario:

$$[\nabla^2 + \lambda_j(\omega)]\{[\Gamma_j C_{11} + \theta_1(C_{21} - C_{11}) + \Pi_j\theta_2(C_{31} - C_{11})]\tilde{\sigma}_{kk} + [-\Gamma_j\theta_1 C_{12} + \theta_1^2(C_{12} - C_{22})$$
$$+ \Pi_j\theta_1\theta_2(C_{12} - C_{32})]\tilde{p}_1 + [-\Gamma_j\theta_2 C_{13} + \theta_1\theta_2(C_{13} - C_{23}) + \Pi_j\theta_2^2(C_{13} - C_{33})]\tilde{p}_2\} = 0. \tag{28}$$

6 Reduction to the decoupled Biot model equations

Dutta and Ode [26] and Berryman [27] have shown that the Biot model equations can be decoupled into two Helmholtz equations whose normal coordinates are complex-valued, frequency-dependent variables. These two Helmholtz equations can be derived from our equations (12) as a special case.

When only one fluid is present in the pore space (i.e., $S_1 = 1$, $K_1 = K_f$, and $S_2 = K_2 = \frac{dS_1}{dp_c} = 0$), the parameters M_1 and M_2 in (A2.4) and (A2.5) become equal to -1 and $\frac{K_f}{\phi}$, respectively. The dimensionless parameters δ_s, δ_1, and δ_2 in (A2.1)–(A2.3) in turn reduce to:

$$\delta_s = \frac{\left(1-\phi-\frac{K_b}{K_s}\right)\frac{\phi K_s}{K_f}}{\left(1-\phi-\frac{K_b}{K_s}+\frac{\phi K_s}{K_f}\right)}, \quad \delta_1 = -\frac{\left(1-\phi-\frac{K_b}{K_s}\right)\phi}{\left(1-\phi-\frac{K_b}{K_s}+\frac{\phi K_s}{K_f}\right)}, \quad \delta_2 = 0. \tag{29.1, 2, 3}$$

As a consequence, the elasticity coefficients a_{ij} in (A1) can be simplified to

$$a_{11} = K_s(1-\phi-\delta_s) = \frac{(1-\phi)\left(1-\phi-\frac{K_b}{K_s}\right)K_s + \frac{\phi K_b K_s}{K_f}}{\left(1-\phi-\frac{K_b}{K_s}+\frac{\phi K_s}{K_f}\right)} = P - \frac{4}{3}G, \tag{30.1}$$

$$a_{12} = a_{21} = -K_s\delta_1 = \frac{\left(1-\phi-\frac{K_b}{K_s}\right)K_s\phi}{\left(1-\phi-\frac{K_b}{K_s}+\frac{\phi K_s}{K_f}\right)} = Q, \tag{30.2}$$

$$a_{22} = (\delta_1+\phi)K_f = \frac{\phi^2 K_s}{\left(1-\phi-\frac{K_b}{K_s}+\frac{\phi K_s}{K_f}\right)} = R, \tag{30.3}$$

$$a_{13} = a_{23} = a_{33} = 0, \tag{30.4}$$

where P, Q, and R are the elasticity coefficients defined by Biot [15, 16]. Therefore, the elasticity coefficients D_{ij} in (6) for a single-fluid system can be written:

$$D_{11} = P + 2Q + R = H, \quad D_{12} = \frac{(Q+R)}{\phi} = C, \tag{31.1, 2}$$

$$D_{22} = \frac{R}{\phi^2} = M, \quad D_{13} = D_{23} = D_{33} = 0, \tag{31.3, 4}$$

where H, C, and M are elasticity coefficients defined by Biot [16]. Accordingly, the elements of matrix $\overline{\overline{B}}$ in (11) become

$$B_{11} = \omega^2(\rho M - \rho_f C)\Delta^{-1}, \quad B_{12} = \omega^2(\rho_f M - qC)\Delta^{-1}, \quad B_{21} = \omega^2(\rho_f H - \rho C)\Delta^{-1}, \tag{32.1, 2, 3}$$

$$B_{22} = \omega^2(qH - \rho_f C)\Delta^{-1}, \quad B_{13} = B_{23} = B_{31} = B_{32} = B_{33} = 0, \tag{32.4, 5}$$

where Δ and q are given by

$$\Delta = MH - C^2, \quad q = \frac{\alpha_s \rho_f}{\phi} + \frac{i\eta_f}{\omega k_s}. \tag{33.1, 2}$$

In (33.2), η_f is the dynamic shear viscosity of the fluid phase. Because only one fluid exists in the pore space, we have $\theta_1 = \phi$ and the relative permeability takes on unit value. Thus, the decoupled Helmholtz equations of the Biot model of poroelasticity can be recovered from our equations (12) as:

$$[\nabla^2 + \lambda_\pm(\omega)]\Phi_\pm(\vec{x}, \omega) = 0, \tag{34}$$

where the eigenvalues $\lambda_\pm(\omega)$ and eigenvectors $\Phi_\pm(\vec{x}, \omega)$ are:

$$\lambda_{\pm}(\omega) = \frac{1}{2}\{(B_{11} + B_{22}) \pm [(B_{11} - B_{22})^2 + 4B_{12}B_{21}]^{\frac{1}{2}}\}, \tag{35.1}$$

$$\Phi_{\pm}(\vec{x}, \omega) = \Gamma_{\pm}\widetilde{e} - \widetilde{\zeta}, \tag{35.2}$$

subject to

$$\Gamma_{\pm} = B_{21}(\lambda_{\pm} - B_{11})^{-1} = (\lambda_{\pm} - B_{22})B_{12}^{-1} = \frac{1}{2B_{12}}\{(B_{11} - B_{22}) \pm [(B_{11} - B_{22})^2 + 4B_{12}B_{21}]^{\frac{1}{2}}\}. \tag{36}$$

The parameter $\widetilde{\zeta}$ is the Fourier transform of the linearized increment of fluid content in a single-fluid system, defined as $\zeta = \phi(e - \varepsilon)$, where ε represents the dilatation of the fluid phase [16]. Equations (34)–(36) are identical to the decoupled Biot [16] model equations obtained by Berryman [27] in the frequency domain. We have recently shown [30] that, when the frequency of wave excitation is much smaller than a critical frequency equal to the kinematic viscosity of the pore fluid divided by the permeability of the porous medium, a time-domain representation of the equations of (34) can be derived which comprises a propagating wave equation and a dissipative wave equation (or telegraph equation), respectively.

7 Conclusions

Poroelasticity is a continuum theory for studying the mechanical behavior of an elastic solid skeleton containing interconnected fluid-saturated pores. When a fluid-filled porous medium is exerted by an applied stress, strong time-dependent coupling occurs between the deformation of the porous material and the fluid flows within it. This attribute leads to model equations describing poroelastic behavior that are in coupled form. Therefore, for the determination of closed-form analytical solutions of the poroelasticity equations under a variety of boundary conditions, it is valuable to perform a normal coordinate transformation that can decouple these equations completely.

A mathematical model for the analysis of the dynamic poroelasticity problem of dilatational wave propagation and attenuation through partially saturated porous media was presented by Lo et al. [13] based on continuum mixture theory, which is general enough to account for three crucial physical interactions arising between immiscible pore fluids: changes in capillary pressure, viscous coupling, and inertial coupling. In the current study, we show that the poroelasticity equations derived in the Lo et al. [13] model can be decoupled into three Helmholtz equations after Fourier transformation. These equations feature three complex-valued frequency-dependent normal coordinates representing independent motional modes, each of which can be expressed in terms of three different linear combinations of the solid dilatation and the linearized increment of each fluid content, or equivalently, three different linear combinations of total dilatational stress and two pore fluid pressures. The choice of which formulation to apply depends on the boundary conditions that are prescribed in engineering applications. As specialized to a saturated porous medium, our Helmholtz equations recover the decoupled Biot [16] model equations as obtained in the frequency domain [27].

Our studies thus have taken significant steps toward yielding a fully decoupled formulation of the poroelastic equations for acoustic wave motions in both single-fluid and two-fluid systems. In future work, we expect to derive a time-domain representation for the two-fluid system at low excitation frequencies with the effect of capillary pressure changes taken into account. As indicated in Table 1, once this goal is realized, any poroelastic response of a homogeneous porous medium permeated by two immiscible, viscous, compressible fluids to suitable boundary conditions can be modeled analytically.

Acknowledgements Gratitude is expressed for financial support to the National Science Council, Taiwan under Contract No. NSC95-2221-E-006-062. The authors also acknowledge financial support for this work partially by National Cheng Kung University under grant number R04 and partially by the Laboratory Directed Research and Development Program at Lawrence Berkeley National Laboratory.

Appendix A: Representation of the coefficients a_{ij}, R_{11}, R_{22}, A_{11}, A_{22}, A_{12}, and A_{21}

According to Lo et al. [13], the elasticity coefficients a_{ij} can be expressed in terms of porosity ϕ and five directly measurable elasticity moduli: the bulk modulus of the porous framework, K_b, the shear modulus of the porous framework, G, and the bulk modulus of the α phase, K_α:

$$a_{11} = K_s(1-\phi-\delta_s), \quad a_{12} = a_{21} = -K_s\delta_1, \quad a_{13} = a_{31} = -K_s\delta_2, \tag{A1.1, 2, 3}$$

$$a_{22} = -\frac{1}{M_1}\left[\left(K_1K_2\frac{\mathrm{d}S_1}{\mathrm{d}p_c} + \frac{K_1K_2S_1}{1-S_1}\frac{\mathrm{d}S_1}{\mathrm{d}p_c} + K_1S_1\right)\delta_1 + \frac{K_1K_2S_1\phi}{1-S_1}\frac{\mathrm{d}S_1}{\mathrm{d}p_c} + K_1S_1\phi\right], \tag{A1.4}$$

$$a_{23} = a_{32} = -\left(\frac{\delta_1\delta_2}{\delta_s}K_s + \frac{K_1K_2\phi}{M_1}\frac{\mathrm{d}S_1}{\mathrm{d}p_c}\right), \tag{A1.5}$$

$$a_{33} = -\frac{1}{M_1}\left\{\left[K_1K_2\frac{\mathrm{d}S_1}{\mathrm{d}p_c} + \frac{K_1K_2(1-S_1)}{S_1}\frac{\mathrm{d}S_1}{\mathrm{d}p_c} + K_2(1-S_1)\right]\delta_2 + \frac{K_1K_2(1-S_1)\phi}{S_1}\frac{\mathrm{d}S_1}{\mathrm{d}p_c} + K_2(1-S_1)\phi\right\}, \tag{A1.6}$$

where S_ξ represents the relative saturation of fluid phase ξ, associated with the volumetric fraction and porosity by $S_\xi = \frac{\theta_\xi}{\phi}$; $p_c = p_1 - p_2$ denotes capillary pressure, a difference between the pressures of the non-wetting and wetting fluids at equilibrium, p_ξ ($\xi = 1, 2$) referring to the (gauge) pressure in each fluid. Equations (A1) reveal that the elasticity coefficients a_{ij} are dependent on the slope, $\frac{\mathrm{d}S_1}{\mathrm{d}p_c}$, of the curve depicting the relation between saturation of the non-wetting fluid and capillary pressure. In the equations of (A1), the parameters δ_s, δ_1, δ_2, M_1, and M_2 are given by

$$\delta_s = \frac{\left(1-\phi-\frac{K_b}{K_s}\right)K_s}{K_s + \frac{M_2}{M_1}\left(\frac{K_b}{K_s}-1+\phi\right)}, \tag{A2.1}$$

$$\delta_1 = \frac{K_1\left(S_1 + K_2\frac{\mathrm{d}S_1}{\mathrm{d}p_c} + \frac{K_2S_1}{1-S_1}\frac{\mathrm{d}S_1}{\mathrm{d}p_c}\right)\left(1-\phi-\frac{K_b}{K_s}\right)}{K_sM_1 + M_2\left(\frac{K_b}{K_s}-1+\phi\right)}, \quad \delta_2 = \frac{K_2\left(1-S_1+\frac{K_1}{S_1}\frac{\mathrm{d}S_1}{\mathrm{d}p_c}\right)\left(1-\phi-\frac{K_b}{K_s}\right)}{K_sM_1 + M_2\left(\frac{K_b}{K_s}-1+\phi\right)}, \tag{A2.2, 3}$$

$$M_1 = -\left(\frac{K_1}{S_1}\frac{\mathrm{d}S_1}{\mathrm{d}p_c} + \frac{K_2}{1-S_1}\frac{\mathrm{d}S_1}{\mathrm{d}p_c} + 1\right), \quad M_2 = \frac{K_1K_2}{\phi S_1(1-S_1)}\frac{\mathrm{d}S_1}{\mathrm{d}p_c} + \frac{K_1S_1}{\phi} + \frac{K_2(1-S_1)}{\phi}, \tag{A2.4,5}$$

where δ_s, δ_1, and δ_2 are dimensionless parameters, and their linear combination with the dilatations of the solid and two immiscible pore fluids can be used to express porosity change caused by a compressional wave traveling through a fluid-filled porous medium: $\Delta\phi = \delta_s e + \delta_1\varepsilon_1 + \delta_2\varepsilon_2$, which is a physical quantity defined as the difference in porosity between the current configuration and a reference configuration, the latter taken here as the unperturbed state prior to wave excitation. The parameters M_1 and M_2 are effective non-wetting fluid storativity factors in response to capillary pressure fluctuations [13]. In view of the generalized Biot–Willis [42] unjacketed experiment for a two-fluid system [13, 22], the physical meaning of the parameter M_1 can be better understood from the relation [see Appendix B]:

$$p_c = \frac{K_1}{M_1}\vec{\nabla}\cdot\vec{u}_1 - \frac{K_2}{M_1}\vec{\nabla}\cdot\vec{u}_2, \tag{A3}$$

where $\vec{u}_\xi$ refers to the displacement vector of fluid phase ξ. Thus, $\frac{K_1}{M_1}$ and $\frac{K_2}{M_1}$ can be, respectively, recognized as the bulk modulus of the non-wetting and wetting fluids in response to changes in capillary pressure when porosity change is held constant.

In reference to the viscous coupling coefficients R_{11} and R_{22} under low-frequency wave excitation, fluid transport through the pore space is of the Poiseuille type, so they can be modeled by the expressions [13, 22, 24, 25]:

$$R_{11} = -\frac{\theta_1^2 \eta_1}{k_s k_{r1}}, \quad R_{22} = -\frac{\theta_2^2 \eta_2}{k_s k_{r2}}, \tag{A4.1,2}$$

where η_ξ is the dynamic shear viscosity of fluid phase ξ; k_s is the intrinsic permeability; and $k_{r\xi}$ is the relative permeability of fluid phase ξ. As noted previously, cross-coupling induced by viscous drag is conventionally neglected in modeling two-phase fluid flows through unsaturated porous media, i.e., $R_{12} = R_{21} = 0$ [18, Chap. 5], [43].

The coefficients A_{11}, A_{22}, A_{12}, and A_{21} describing the effect of inertial coupling are related to fluid properties and pore structure by [13, 25]:

$$A_{11} = (1-\alpha_s)\rho_1\theta_1, \quad A_{22} = (1-\alpha_s)\rho_2\theta_2, \quad A_{12} = A_{21} = -0.1\sqrt{\alpha_s^2 \rho_1 \rho_2 \theta_1 \theta_2}, \tag{A5.1, 2, 3}$$

where α_s is a geometrical factor independent of solid or fluid densities to characterize how the structure of the porous medium restricts the flow of fluid, commonly termed tortuosity [1, 44, 45]. In principle, α_s can be determined by acoustic or electrical measurements [46]; a theoretical value of $\alpha_s = \frac{1}{2}(1 + \frac{1}{\phi})$ was given by Berryman [45] for a fluid-containing porous medium whose solid grains are spherical. The cross-coupling coefficients A_{12} and A_{21} are typically considered to be symmetric, i.e., $A_{12} = A_{21}$ [13, 24, 25].

Appendix B: Relation of M_1 to capillary pressure p_c

By definition, capillary pressure is equal to the difference between the non-wetting and wetting fluid pressures:

$$p_c = p_1 - p_2. \tag{B1}$$

Thus, changes in capillary pressure can be written as

$$\frac{\partial p_c}{\partial t} = \frac{\partial p_1}{\partial t} - \frac{\partial p_2}{\partial t}. \tag{B2}$$

Following Lo et al. [13], the balance equations of mass applied to the non-wetting and wetting fluids take the form:

$$\frac{\partial p_1}{\partial t} = -\frac{K_1}{S_1}\frac{\partial S_1}{\partial t} - \frac{K_1}{\phi}\frac{\partial \phi}{\partial t} - K_1 \vec{\nabla} \cdot \vec{v}_1, \tag{B3.1}$$

$$\frac{\partial p_2}{\partial t} = -\frac{K_2}{(1-S_1)}\frac{\partial (1-S_1)}{\partial t} - \frac{K_2}{\phi}\frac{\partial \phi}{\partial t} - K_2 \vec{\nabla} \cdot \vec{v}_2, \tag{B3.2}$$

where $\vec{v}_\xi$ and K_ξ denote the velocity vector and bulk modulus of fluid phase ξ, respectively. Substituting equations (B3) to eliminate $\frac{\partial p_1}{\partial t}$ and $\frac{\partial p_2}{\partial t}$ in equation (B2), one obtains

$$\begin{aligned}\frac{\partial p_c}{\partial t} &= \frac{\partial p_1}{\partial t} - \frac{\partial p_2}{\partial t} = -\frac{K_1}{S_1}\frac{\partial S_1}{\partial t} - \frac{K_1}{\phi}\frac{\partial \phi}{\partial t} - K_1 \vec{\nabla} \cdot \vec{v}_1 - \frac{K_2}{(1-S_1)}\frac{\partial S_1}{\partial t} + \frac{K_2}{\phi}\frac{\partial \phi}{\partial t} + K_2 \vec{\nabla} \cdot \vec{v}_2 \\ &= -\left[\frac{K_1}{S_1} + \frac{K_2}{(1-S_1)}\right]\frac{\partial S_1}{\partial t} - \frac{(K_1 - K_2)}{\phi}\frac{\partial \phi}{\partial t} - K_1 \vec{\nabla} \cdot \vec{v}_1 + K_2 \vec{\nabla} \cdot \vec{v}_2.\end{aligned} \tag{B4}$$

Under the isothermal conditional, the capillary pressure can be uniquely described by the function $p_c = p_c(S_1)$ if the hysteresis effect is neglected. Thus, changes in relative saturation of the non-wetting fluid can be expressed as

$$\frac{\partial S_1}{\partial t} = \frac{\mathrm{d}S_1}{\mathrm{d}p_c}\frac{\partial p_c}{\partial t}. \tag{B5}$$

Combination of (B4) and (B5) leads to

$$\frac{\partial p_c}{\partial t} = \frac{(K_1 - K_2)}{M_1 \phi}\frac{\partial \phi}{\partial t} + \frac{1}{M_1}(K_1 \vec{\nabla} \cdot \vec{v}_1 - K_2 \vec{\nabla} \cdot \vec{v}_2). \tag{B6}$$

In the generalized Biot–Willis [42] unjacketed experiment for a two-fluid system, one of the physical constraints imposed in the experiment is that porosity remains unchanged [13, 22]:

$$\frac{\partial \phi}{\partial t} = 0. \tag{B7}$$

It follows that (B6) can be reduced to

$$\frac{\partial p_c}{\partial t} = \frac{K_1}{M_1}\vec{\nabla} \cdot \vec{v}_1 - \frac{K_2}{M_1}\vec{\nabla} \cdot \vec{v}_2. \tag{B8}$$

After time integration, the linearized form of (B8) can be formulated as

$$p_c = \frac{K_1}{M_1}\vec{\nabla} \cdot \vec{u}_1 - \frac{K_2}{M_1}\vec{\nabla} \cdot \vec{u}_2. \tag{B9}$$

References

1. Johnson DL (1986) Recent developments in the acoustic properties of porous media. In: Sette D (ed) Proceedings of the international school of physics Enrico Fermi Course XCIII, Frontiers in physical acoustics. North Holland, Amsterdam, pp 255–290
2. Finol A, Farouq Ali SM (1975) Numerical simulation of oil production with simultaneous ground subsidence. J Soc Petrol Eng 259:411–422
3. Kosloff D, Scott RF, Scranton M (1980) Finite element simulation of Wilmington oil field subsidence: I. linear modeling. Tectonophys 65:339–368. doi:10.1016/0040-1951(80)90082-7
4. Corapcioglu MY, Bear J (1983) A mathematical model for regional land subsidence due to pumping: 3 integrated equations for a phreatic aquifers. Water Resour Res 19:895–908. doi:10.1029/WR019i004p00895
5. Beresnev IA, Johnson PA (1994) Elastic-wave stimulation of oil production: a review of methods and results. Geophysics 59(6):1000–1017. doi:10.1190/1.1443645
6. Spanos T, Davidson B, Dusseault M, Shand D, Samaroo M (2003) Pressure pulsing at the reservoir scale: a new IOR approach. J Can Petrol Technol 42(2):16–28
7. Montgomery DR, Manga M (2003) Steamflow and water well response to earthquakes. Science 300:2047–2049. doi:10.1126/science.1082980
8. Gambolati G, Teatini P, Baú D, Ferronato M (2000) Importance of poroelastic coupling in dynamically active aquifers of the Po river basin, Italy. Water Resour Res 36(9):2443–2459. doi:10.1029/2000WR900127
9. Wang HF (2000) Theory of linear poroelasticity with applications to geomechanics and hydrogeology. Princeton University Press, Princeton
10. Li X, Zhong L, Pyrak-Nolte LJ (2001) Physics of partially saturated porous media: residual saturation and seismic-wave propagation. Annu Rev Earth Planet Sci 29:419–460. doi:10.1146/annurev.earth.29.1.419
11. Cosenza P, Ghoreychi M, de Marsily G, Vasseur G, Violette S (2002) Theoretical prediction of poroelastic properties of argillaceous rocks from in situ specific storage coefficient. Water Resour Res 38(10):1207. doi:10.1029/2001WR001201
12. Coussy O (2004) Poromechanics. Wiley, Chichester
13. Lo WC, Sposito G, Majer E (2005) Wave propagation through elastic porous media containing two immiscible fluids. Water Resour Res 41:W02025. doi:10.1029/2004WR003162
14. Biot MA (1941) General theory of three-dimensional consolidation. J Appl Phys 12:155–164. doi:10.1063/1.1712886
15. Biot MA (1956) Theory of propagation of elastic waves in a fluid-saturated porous solid, I. Low-frequency range, II. Higher frequency range. J Acoust Soc Am 28(2):168–191. doi:10.1121/1.1908239
16. Biot MA (1962) Mechanics of deformation and acoustic propagation in porous media. J Appl Phys 33(4):1482–1498. doi:10.1063/1.1728759
17. Drew DA, Passman SL (1999) Theory of multicomponent fluids. Springer-Verlag, New York
18. Dullien FLA (1992) Porous media: fluid transport and pore structure. Academic Press, San Diego
19. Brutsaert W (1964) The propagation of elastic waves in unconsolidated unsaturated granular mediums. J Geophys Res 69(2):243–257. doi:10.1029/JZ069i002p00243
20. Brutsaert W, Luthin JM (1964) The velocity of sound in soils near the surface as a function of the moisture content. J Geophys Res 69(4):643–652. doi:10.1029/JZ069i004p00643
21. Garg SK, Nayfeh AH (1986) Compressional wave propagation in liquid and/or gas saturated elastic porous media. J Appl Phys 60(9):3045–3055. doi:10.1063/1.337760
22. Tuncay K, Corapcioglu MY (1997) Wave propagation in poroelastic media saturated by two fluids. J Appl Mech 64(2):313–320. doi:10.1115/1.2787309
23. Wei C, Muraleetharan KK (2002) A continuum theory of porous media saturated by multiple immiscible fluids: I. Linear poroelasticity. Int J Eng Sci 40:1807–1833. doi:10.1016/S0020-7225(02)00068-X

24. Berryman JG, Thigpen L, Chin RCY (1988) Bulk elastic wave propagation in partially saturated porous solids. J Acoust Soc Am 84(1):360–373. doi:10.1121/1.396938
25. Santos JE, Corbero JM, Douglas J (1990) Static and dynamic behavior of a porous solid saturated by a two-phase fluid. J Acoust Soc Am 87(4):1428–1438. doi:10.1121/1.399439
26. Dutta NC, Ode H (1979) Attenuation and dispersion of compressional waves in fluid-filled porous rocks with partial gas saturation (White model)—Part I: Biot theory. Geophysics 11:1777–1788. doi:10.1190/1.1440938
27. Berryman JG (1983) Dispersion of extensional waves in fluid-saturated porous cylinders at ultrasonic frequencies. J Acoust Soc Am 74(6):1805–1812. doi:10.1121/1.390266
28. Chandler RN, Johnson DL (1981) The equivalence of quasistatic flow in fluid-saturated porous media and Biot's slow wave in the limit of zero frequency. J Appl Phys 52(5):3391–3395. doi:10.1063/1.329164
29. Lo WC, Sposito G, Majer E (2002) Immiscible two-phase fluid flows in deformable porous media. Adv Water Resour 25(8–12):1105–1117
30. Lo WC, Sposito G, Majer E (2006) Low-frequency dilatational wave propagation through fully-saturated poroelastic media. Adv Water Resour 29(3):408–416. doi:10.1016/j.advwatres.2005.05.012
31. Lo WC (2006) Decoupling of the coupled poroelastic equations for quasistatic flow in deformable porous media containing two immiscible fluids. Adv Water Resour 29(12):1893–1900. doi:10.1016/j.advwatres.2006.01.002
32. Lo WC, Sposito G, Majer E (2007) Low-frequency dilatational wave propagation through unsaturated porous media containing two immiscible fluids. Transp Porous Media 68:91–105. doi:10.1007/s11242-006-9059-2
33. de la Cruz V, Spanos TJT (1985) Seismic wave propagation in a porous medium. Geophysics 50:1556–1565. doi:10.1190/1.1441846
34. Pride SR, Gangi AF, Morgan FD (1992) Deriving the equations of motion for porous isotropic media. J Acoust Soc Am 92(6):3278–3290. doi:10.1121/1.404178
35. Maas C (1986) The use of matrix differential calculus in problems of multiple-aquifer flow. J Hydrol (Amst) 88:43–67. doi:10.1016/0022-1694(86)90196-4
36. van Genuchten MT (1980) A closed-form equation for predicting the hydraulic conductivity of unsaturated soils. Soil Sci Soc Am J 44(5): 892–898
37. Mualem Y (1976) A new model for predicting the hydraulic conductivity of unsaturated porous media. Water Resour Res 12(3):513–522. doi:10.1029/WR012i003p00513
38. Das BM (1993) Principles of soil dynamics. PWS-Kent, Boston
39. Bishop AW (1959) The principle of effective stress. Teknisk Ukeblad 39:859–863
40. Bear J, Bachmat Y (1991) Introduction to modeling of transport phenomena in porous media. Kluwer, Netherlands
41. Whitaker S (1973) Transport equations for multiphase system. Chem Eng Sci 28(1):139–147. doi:10.1016/0009-2509(73)85094-8
42. Biot MA, Willis DG (1957) The elastic coefficients of the theory of consolidation. J Appl Mech 24:594–601
43. Gray WG (1983) General conservation equations for multi-phase systems: 4. Constitutive theory including phase change. Adv Water Resour 6:130–140. doi:10.1016/0309-1708(83)90025-8
44. Stoll RD (1974) Acoustic waves in saturated sediments. In: Hampton L (ed) Physics of sound in marine sediments. Plenum, New York, pp 19–39
45. Berryman JG (1980) Confirmation of Biot's theory. Appl Phys Lett 37(4):382–384. doi:10.1063/1.91951
46. Johnson DL, Plona TJ, Scala C, Pasierb F, Kojima H (1982) Tortuosity and acoustic slow waves. Phys Rev Lett 49(25):1840–1844. doi:10.1103/PhysRevLett.49.1840